On the Roots of the Hypergeometric and Bessel's Functions.

By M. B. Porter.

Introduction.

The questions discussed in this paper relate primarily to the real roots of Bessel's functions of negative order, the real roots of the convergents of the continued fraction for J_n/J_{n+1}, and the real roots of the hypergeometric series. The theorems obtained are applied to the problems of enumerating the imaginary roots of $J_n(x)$ and the roots of $F(\alpha, \beta, \gamma, x)$ between 0 and 1.

The methods employed throughout are those developed by Sturm in his first Memoir, on the real roots of functions defined by homogeneous linear differential equations of the second order, published in the first volume of Liouville's Journal. The theorems of the above-mentioned memoir, as enunciated by Sturm, are applicable only to intervals of the x-axis, the end points being included, in which there are no singular points of the differential equation. It is found, however, that in certain cases these theorems admit of a generalization to the case in which one or both of the ends of the interval in question are singular points of the differential equation. These generalized forms of Sturm's theorem are especially useful in dealing with the roots of the principal branches of the hypergeometric and Bessel's functions.

The following theorems of Sturm's memoir, which for convenience of reference will be stated at this point, are of constant application:

I. *p and q being continuous real functions of the real variable x, between two successive roots of a solution of the differential equation*

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

*will lie one and only one root of any linearly independent solution.**

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*Liouville, tom. I, p. 125.

A somewhat generalized form of this theorem is the following:

I'. *If, within an interval ab (excluding the ends) the coefficients of the differential equation*

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

*are continuous, and if there exists a solution y_1 which does not vanish between a and b and such that its ratio to a linearly independent solution y_2 approaches zero as we approach each end of the interval, then y_2 vanishes once and only once between a and b.**

In the case of a regular singular point at a, y_1 would be the solution corresponding to the larger exponent of a.

II. *If, ϕ_1 and ϕ_2 being continuous real functions of the real variable x, $\phi_1 < \phi_2$, and if y_1 and y_2 both vanish at a point x_0 and satisfy respectively the equations*

$$\frac{d^2y}{dx^2} = \phi_1(x)\,y, \qquad [1]$$

$$\frac{d^2y}{dx^2} = \phi_2(x)\,y, \qquad [2]$$

then if y_2 has n roots to the right (left) of x_0, y_1 will have at least n roots there and the x^{th} root of y_2 will be greater (less) than the x^{th} root of y_1 from x_0.†

III. *In a certain interval $\alpha\beta$ of the x-axis, if $\phi_1(x)$ and $\phi_2(x)$ are continuous, and $\phi_1(x) < \phi_2(x)$, then between two successive*‡ *roots, lying in this interval of a solution of the equation*

$$\frac{d^2y}{dx^2} = \phi_1(x)\,y, \qquad [1]$$

there cannot lie more than one root of a solution of the equation§

$$\frac{d^2y}{dx^2} = \phi_2(x)\,y. \qquad [2]$$

Here it is to be noted that the continuity of ϕ_1 and ϕ_2 restricts the application of the theorem to an interval of the x-axis containing no singular point of either

* Math. Bull., Mar. 1897, p. 211 and footnote.

† Liouville, tom. I, p. 125.

‡ The ends of an interval $x_{\nu-1}x_\nu$ determined by two consecutive roots are not excluded. Cf. II, above.

§ Liouville, tom. I, p. 136.

[1] or [2], although α or β might be singular points of [1] or [2]. A slight generalization of III, which will be useful, is the following:

III′. Let y_1 be any solution of [1] and x_1 be the root of y_1 which in the interval $\alpha\beta$ is nearest α, then *no solution of* [2] *can vanish more than once between* x_1 *and* α. This follows directly from I and II, for let us suppose that y_2, a solution of [2], did vanish more than once between x_1 and α, then by I, a solution $\bar{y}_2$ of [2], which vanishes at x_1, will have at least one root between x_1 and α, and by II so must y_1, which contradicts our hypothesis that x_1 is the root of y_1 nearest α in the interval $\alpha\beta$.

On the Roots of Bessel's Functions.

The first questions that will be considered relate to the real roots of the transcendental function $J_n(x)$, where it will be supposed that the index n is real. The case where the index is positive has already been treated in considerable detail in the Math. Bulletin, Mar. 1897, in the paper by Professor Bôcher, "On Certain Methods of Sturm and their Application to the Roots of Bessel's Functions."* We shall accordingly consider mainly the case where n is negative.

$J_n(x)$ is a particular solution of the homogeneous linear differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0,$$

and is defined by the series†

$$J_n(x) = \frac{x^n}{2^n\Gamma(1+n)}\left[1 - \frac{1}{1!\,(1+n)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,(1+n)(2+n)}\left(\frac{x}{2}\right)^4 - \right]$$

except when n is integral we have the linearly independent solution

$$J_{-n}(x) = \frac{x^{-n}}{2^{-n}\Gamma(1-n)}\left[1 - \frac{1}{1!\,(1-n)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,(1-n)(2-n)}\left(\frac{x}{2}\right)^4 - \right],$$

the general solution being, when n is not integral, of the form

$$y_n = aJ_n(x) + bJ_{-n}(x).$$

When n is an integer we have the important relation

$$J_{-n}(x) = (-1)^n J_n(x).$$

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* This paper will be referred to in future by Math. Bull., Mar. 1897.

† For other definitions, e. g. by complex integrals, see Gray and Mathews' "Bessel Functions," p. 59.

Sturm deduced, by the methods developed in the memoir already referred to, the well-known theorem that $J_n(x)$ has, when n is real, an infinite number of real roots.

The series for $J_n(x)$ shows that the negative roots are numerically equal to the positive roots, so that in future we shall only speak of the latter.

§1.—*On the Variation of the Roots of $J_n(x)$.*

In the 10th vol. of the Math. Ann. (footnote, p. 137), Schläfli has shown that the positive roots of $J_n(x)$, n positive, increase with n.* When n is negative, the corresponding theorem is: *the positive roots of $J_n(x)$, $n<0$, decrease with n.* Both statements can, of course, be embodied in the single theorem: *the positive roots of $J_n(x)$ decrease as n decreases.* To prove this, when n is negative we proceed as follows: The large roots of $J_n(x)$ are given approximately by the expression $x_0=\frac{\pi}{4}(2n+1+4p)$,† p taking on in succession all large integral values; so that if $n<0$ decrease, x_0 will decrease. It is only necessary to note that $y=\sqrt{x}J_n(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2}=\left(\frac{4n^2-1}{x^2}-1\right)y=\phi(x,n)\,y,$$

and that $\phi(x,n)$, $n<0$, as n decreases, increases for all values of x. Thus when $x_0>0$ decreases, all the smaller positive roots must decrease, for otherwise we should have two roots $J_{n-\kappa}(x)$ $(\kappa>0)$ between two consecutive roots of $J_n(x)$, which by III is impossible.

Since, as n increases, the roots of $J_n(x)$ and $J_{-n}(x)$ are moving in opposite directions, they must pass each other, and this takes place when and only when n passes through an integral value; for it is only then that J_n and J_{-n} cease to be linearly independent of each other. Theorem III′ tells us that, *if x_1 denote the smallest positive root of $J_n(x)$, $n<0$, $J_{n-\kappa}(x)$ $(\kappa>0)$ cannot have more than one root between 0 and x_1.*

Let us now suppose that n, starting with the value $-\frac{1}{2}$, in which case

$$J_n(x)=\sqrt{\frac{2}{\pi x}}\cos x$$

decrease, the positive roots of $J_n(x)$ will decrease.

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* For a proof of this theorem see p. 203. Another deduction of this theorem, which I shall extend to the case where n is negative, is given on page 212, Math. Bull., Mar. 1897.

† Gray and Mathews, "Bessel Functions," p. 40.

The curve $y = J_n(x)$, to the right of the origin, will consist of an infinite number of arches which, as x increases, become flatter and flatter while near the origin, the curve has an infinite branch going off to infinity in the positive direction. When n passes through the value -1, the Γ-function in the expansion for $J_n(x)p$ changes sign and the branch at the origin goes off to infinity in the opposite direction. Thus it is clear that when n decreased from the value $-\frac{1}{2}$ to the value $-1-\varepsilon$, where ε is a small positive quantity, an odd number of roots disappeared at the origin. The positive roots of $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$ are: $x_1 = \frac{\pi}{2}$, $x_2 = \frac{3\pi}{2} \ldots . - x_\kappa = \frac{2\kappa - 1}{2}\pi \ldots .$ The interval $0, \frac{\pi}{2}$ is by III′ the first to become vacant, the root x_1 decreasing from the value $\frac{\pi}{2}$ to 0 while the root x_2 remains in the interval $\frac{\pi}{2}, \frac{3\pi}{2}$, so that but one positive root was lost when n decreased through -1. In the same way it is shown that *when n decreases through any negative integer, one positive root is lost at the origin and only one.* The negative roots of $J_n(x)$ being equal in absolute value to the positive roots, behave precisely like them, and consequently when n decreases through a negative integer, one and only one negative root is lost at the origin. Denoting by x_n the n^{th} positive root of $J_n(x)$, $n < 0$, a question that naturally presents itself is: how large can we let $-\varkappa$ ($\varkappa < 0$) become and $J_{n+\kappa}(x)$ still have a single root in each of the intervals $0x_1$, x_1x_2, x_2x_3, etc. The preceding considerations enable us to answer this question at once. *If n and $\varkappa$ are negative,* the positive roots of $J_n(x)$ and $J_{n+\kappa}(x)$ will separate each other if $-\varkappa < E(-n+1)+n$.*

§2.—*Determination of Intervals for Roots of $J_n(x)$.*

The theorems relative to the motion of the real roots of $J_n(x)$ as n decreases can be applied to some interesting questions concerning the distribution of these roots.

Bessel† showed that if we divide the axis of reals up into intervals of $\frac{\pi}{2}$, the first interval to the right of the origin being $0, \frac{\pi}{2}$, then the roots of $J_0(x)$

*If n and κ are positive, the corresponding theorem (Math. Bull., Mar. 1897, p. 212) is: If $\kappa \leqq 2$ *the positive roots of $J_n(x)$ and $J_{n+\kappa}(x)$ separate each other.*

† Berlin. Abhandlungen (1824), §14, "Untersuchung des Theils der Planetarischen Störungen," etc.

all lie in the even intervals to the right and left of the origin, in each of which there is an odd number of roots. Neumann,* slightly modifying Bessel's proof, shows that $J_n(x)$, $-\frac{1}{2} < n < \frac{1}{2}$, has an odd number of roots in the intervals in question. We have seen that if $\frac{2i-1}{2} < n < \frac{2i+1}{2}$, where it will be supposed that i is a positive integer or 0, the $\varkappa^{\text{th}}$ positive root of $J_n(x)$ lies between the $\varkappa^{\text{th}}$ positive roots of $J_{\frac{2i-1}{2}}(x)$ and $J_{\frac{2i+1}{2}}(x)$, and that $J_n(x)$ has but one root in this interval. When $i = 0$, we have: *The $\varkappa^{th}$ root of $J_n(x)$, $-\frac{1}{2} < n < \frac{1}{2}$, lies between the $\varkappa^{th}$ root of $J_{-\frac{1}{2}}(x)$ and of $J_{\frac{1}{2}}(x)$. Since $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$ and $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$, the $\varkappa^{th}$ root of $J_n(x)$ lies between $\frac{2\varkappa - 1}{2}\pi$ and $\varkappa\pi$,* i. e. *in the $\varkappa^{th}$ even quadrant.*

When $i = 1$, we have: The $\varkappa^{\text{th}}$ root of $J_n(x)$, $\frac{1}{2} < n < \frac{3}{2}$, lies between $\varkappa\pi$ and the $\varkappa^{\text{th}}$ root of $J_{\frac{3}{2}}(x)$,

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right),$$

the first positive root of $J_{\frac{3}{2}}(x)$ is greater than π, the first positive root of $J_{\frac{1}{2}}(x)$, and it is readily seen $\left(\text{by substituting in } J_{\frac{3}{2}}(x) \text{ the values } x = \frac{\pi}{2}, \frac{3\pi}{2}; \pi, 2\pi, \text{etc.}\right)$ that the $\varkappa^{\text{th}}$ root of $J_{\frac{3}{2}}(x)$ lies in the $\varkappa + 1^{\text{st}}$ odd quadrant. *Thus the $\varkappa^{th}$ positive root of $J_n(x)$, $\frac{1}{2} < n < \frac{3}{2}$, lies in the $\varkappa + 1^{st}$ odd quadrant.*

In the same way we could go on determining comparatively narrow limits for the roots of $J_n(x)$, $n > \frac{3}{2}$, by determining the intervals of $\frac{\pi}{2}$ in which the roots of $J_{i+\frac{1}{2}}(x)$ lie. Unfortunately there seems to be no simple law governing the distribution† of the smaller roots of $J_{i+\frac{1}{2}}(x)$.

Very good approximations to the roots of $J_0(x)$, which we have seen lie one by one‡ in the even quadrants to the right and left of the origin, are given by the mid-points of the intervals in question.

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* "Theorie der Bessel'sche Functionen," pp. 65–6.

† M. P. Rudski (tome XVIII of the Mém. de la Soc. Royale de Liège), after a somewhat cumbrous analysis arrives at the result, which is readily seen from the tables to be erroneous, that the κ^{th} root of $J_{i+\frac{1}{2}}(x)$ (i integral) lies between $(i + 2\kappa - 1)\frac{\pi}{2}$ and $(i + 2\kappa)\frac{\pi}{2}$.

‡ A fact noticed in the review of Gray and Mathews' "Bessel Functions," Math. Bull., May, 1896, p. 258.

§3.—*On the Roots of* $J_n(x)$ *and* $J_{n+i}(x)$.

When i is integral, and in this section we shall suppose it so, $J_n(x)$ and $J_{n+i}(x)$ are kindred (*verwandt*) functions and relations of the form*

$$J_{n+i}(x) = g_{i-1}^{(n)}(x) J_{n+1}(x) - g_{i-2}^{(n)}(x) J_n(x)$$

exist where the g's are rational integral functions of $\frac{2}{x}$. Such relations are only a limiting case of a general class of similar relations connecting three kindred hypergeometric functions.

The roots† of these rational functions play an important rôle in the study of the roots of $J_n(x)$. The first case to be considered is where $i = \pm 1$, and here we have the theorem:‡ *The positive roots of* $J_{n\pm 1}(x)$ *and* $J_n(x)$ *separate each other.* This follows at once from the two formulæ

$$\frac{d}{dx}\{x^{-n}J_n(x)\} = -x^{-n}J_{n+1}(x), \tag{1}$$

$$\frac{d}{dx}\{x^{n+1}J_{n+1}(x)\} = x^{n+1}J_n(x). \tag{2}$$

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* Gray and Mathews' "Bessel Functions," p. 13.

† Hurwitz, "Ueber die Nullstellen der Bessel'sche Function," Math. Ann., Bd. 33 (1888).

‡ Van Vleck (vol. XIX, p. 75, American Journal) first established this theorem by means of the relation

$$J_{n+1}J_{-n} - J_{-n-1}J_n = \frac{2\sin(n+1)\pi}{\pi x} \left(n \text{ integral } J_{n+1}Y_n - J_nY_{n+1} = \frac{1}{x}\right),$$

extending the method to the contiguous P-functions of Riemann by means of the analogous relation connecting two pairs of linearly independent contiguous P-functions. In the Math. Bull., Mar. 1897, this theorem is proved by methods more analogous to those here used. Van Vleck's method can be applied to similar questions concerning the real roots of any solution y of the homogeneous linear differential equation

$$y'' + py' + qy = 0, \tag{[1]}$$

and the real roots of y' and y''. Here we make use of Abel's relation (α) $y_1y_2' - y_2y_1' = -Ae^{-\int p dx}$, and its derivative ($\beta$) $y_1y_2'' - y_2y_1'' = -Ape^{-\int p dx}$, where y_1 and y_2 denote linearly independent solutions of [1]. The functions y_1' and y_1'' satisfy a homogeneous linear differential equation of the second order with the singular points of [1] and, in general, certain accessory singular points besides. Equations (α) and (β), applied to Bessel's equation, show that *if* $|x| > |n|$ *the real roots of either* $J_n'(x)$ *or* $J_n''(x)$ *and of* $J_n(x)$ *separate each other.* This theorem was given in the Math. Bull., Mar. 1897, p. 207. It is to be noted that the theorems thus obtained do not, in general, relate to corresponding branches of contiguous functions. Applied to any solution y of the hypergeometric differential equation

$$y'' - \frac{\gamma - (\alpha+\beta+1)x}{x(1-x)}y' - \frac{\alpha\beta}{x(1-x)}y = 0 \tag{[2]}$$

(α) yields: *In any interval of the x-axis containing no singular point of* (2), *the roots of* $\frac{d^ry}{dx^r}$ *and* $\frac{d^{r+1}y}{dx^{r+1}}$ *separate each other.* And in the same way, *the roots of* $P_m^n(x)$ *and* $P_m^{n+1}(x)$ *between* $+1$ *and* -1 *separate each other* ($P_m^n(x)$ denoting the associated function of Spherical Harmonics).

(1) by Rolle's theorem shows that between two consecutive positive roots of $J_n(x)$ lies at least one root of $J_{n+1}(x)$, and (2) that between two consecutive positive roots of $J_{n+1}(x)$ lies at least one root of $J_n(x)$, hence the theorem. In the same way it can be shown that the positive roots of

$$y_n = a\,J_n(x) + bJ_{-n}(x) \tag{3}$$

and of

$$-aJ_{n+1}(x) + bJ_{-n-1}(x) \tag{4}$$

separate each other. It is to be noted that (1) and (2) are not contiguous unless a or b vanish or n is integral.

Attention has already been called to the relation

$$J_{n+i}(x) = g_{i-1}^{(n)}(x)\,J_{n+1}(x) - g_{i-2}^{(n)}(x)\,J_n(x).$$

The following lemma of constant application will show the use that can be made of the roots of $g_{i-1}^{(n)}(x)$ in determining the relative positions of the roots of $J_n(x)$ and $J_{n+i}(x)$. Let x_j and x_{j+1} be two consecutive positive roots of $J_n(x)$, we have

$$J_{n+i}(x_j) \quad = g_{i-1}^{(n)}(x_j)\,J_{n+1}(x_j)$$

and

$$J_{n+i}(x_{j+1}) = g_{i-1}^{(n)}(x_{j+1})\,J_{n+1}(x_{j+1}).$$

Since neither x_j nor x_{j+1} is zero, neither $J_{n+1}(x_j)$ nor $J_{n+1}(x_{j+1})$ can vanish. We know that $J_{n+1}(x)$ changes sign once between x_j and x_{j+1}, therefore unless $g_{i-1}^{(n)}(x)$ change sign an odd number of times between x_j and x_{j+1}, $J_{n+i}(x)$ will have an odd number of roots between x_j and x_{j+1}. If $|h+i| > (n)$, $J_{n+i}(x)$ can have at most one root between x_j and x_{j+1}, by theorem III of Sturm, p. 194. Thus from the relations,

$$J_{n+2}(x) = \frac{2(n+1)}{x}J_{n+1}(x) - J_n(x), \qquad n > 0$$

$$J_n(x) \quad = \frac{2(n-1)}{x}J_{n-1} - J_{n-2}(x), \qquad n < 0$$

we get the general theorem: *The positive roots of $J_{n\pm 2}(x)$ and $J_n(x)$ separate each other.*[*]

§4.—*On the Functions $G_i^{(n)}(x)$.*

The rational functions that we have denoted above by $g_{i-1}^{(n)}(x)$ and $g_{i-2}^{(n)}(x)$ are computed at once from the relations *inter contiguas.* We have

* Math. Bull., Mar. 1897, p. 207.

$$J_{n+i}(x) = \frac{2(n+i-1)}{x} J_{n+i-1}(x) - J_{n+i-2},$$

$$J_{n+i-1} = \frac{2(n+i-2)}{x} J_{n+i-2} - J_{n+i-3},$$

$$J_{n+i-2} = \frac{2(n+i-3)}{x} J_{n+l-i} - J_{n+i-4},$$

. . . .

$$J_{n+3} \quad = \frac{2(n+2)}{x} J_{n+2} - J_{n+1},$$

$$J_{n+2} \quad = \frac{2(n+1)}{x} J_{n+1} - J_{n}.$$

Thus
$$J_{n+i} \quad = g_{\varkappa}^{(n)} J_{n+i-\varkappa} - g_{\varkappa-1}^{(n)}(x) J_{n+i-\varkappa-1}$$
$$= g_{i-1}^{n}(x) J_{n+1} - g_{i-2}^{n}(x) J_{n},$$

where
$$g_{\varkappa}^{(n)}(x) = \frac{2(n+i-\varkappa)}{x} g_{i-1}^{(n)}(x) - g_{i-2}^{(n)}(x) \qquad [I]$$

and
$$g_{-1}^{(n)}(x) = 0, \quad g_{0}^{(n)}(x) = 1,$$

thus $g_{\varkappa}^{(n)}(x)$ is a polynomial in $\frac{2}{x}$ of degree $\varkappa$.

From the recurring relation I we get

$$g_{1}^{(n)}(x) = \frac{2(n+i-1)}{x},$$

$$g_{2}^{(n)}(x) = \frac{2^2}{x^2}(n+i-2)(n+i-1) - 1,$$

$$g_{3}^{(n)}(x) = \frac{2^3}{x^3}(n+i-3)(n+i-2)(n+i-1) - \frac{2}{x}\, 2(n+i-2),$$

and by inspection,

$$g_{\varkappa}^{(n)}(x) = \frac{2^{\varkappa}}{x^{\varkappa}}(n+i-\varkappa)(n+i-\varkappa+1) \ldots . (n+i-2)(n+i-1),$$
$$- \frac{2^{\varkappa-2}}{x^{\varkappa-2}} \frac{(\varkappa-1)}{1\,!}(n+i-\varkappa+2) \ldots . (n+i-2),$$
$$+ \frac{2^{\varkappa-4}}{x^{\varkappa-4}} \frac{(\varkappa-2)(\varkappa-3)}{2\,!}(n+i-\varkappa+3) \ldots . (n+i-3),$$

. . . .

the r^{th} term being

$$+(-1)^{r-1}\frac{2^{\kappa-2r+2}}{x^{\kappa-2r+2}}\frac{(\kappa+1-r)(\kappa-r)\dots(\kappa-2r+3)}{(r-1)!}(n+i-\kappa+r-1)\dots(n+i-r).$$

Since the expression $\dfrac{(\kappa+1-r)(\kappa-r)\dots\dots(\kappa-2r+3)}{(r-1)!}$ is always a positive integer, the sign of the r^{th} term depends only on the factor

$$(n+i-\kappa+r-1)\dots\dots(n+i-r),$$

moreover, the coefficients of the various powers of $\dfrac{2}{x}$ are rational integral functions of n and are therefore rational numbers if n is rational.

$$J_{n+i}(x)=g_{i-1}^{(n)}(x)\,J_{n+1}(x)-g_{i-2}^{(n)}(x)\,J_n(x).$$

If $J_n(x)$ and $J_{n+i}(x)$ both vanish for the same value of x as $x_0\neq 0$, we must have $g_{i-1}^{(n)}(x_0)=0$,* i.e. if n is rational or merely algebraically irrational, the common root x_0 must be algebraically irrational. If we knew that the positive roots of $J_n(x)$ were, when n is rational or algebraically irrational, transcendental numbers, we could at once conclude that $J_n(x)$ and $J_{n+i}(x)$ have no positive root in common.†

When n is the half of an odd integer it is easy to show that the roots of $J_n(x)$ are transcendental numbers.

$$J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sin x,$$

$$J_{\frac{3}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left\{\frac{\sin x}{x}-\cos x\right\},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$J_{i+\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left\{g_i^{(i+\frac{1}{2})}(x)\sin x-g_{i-1}^{(i+\frac{1}{2})}(x)\cos x\right\}.$$

The coefficients of the various powers of $\dfrac{2}{x}$ in the g-functions are rational numbers, and it is clear from the relation (I), p. 201, that $g_\kappa^{(n)}(x)$ and $g_{\kappa-1}^{(n)}(x)$ can-

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* Since $J_n(x)$ and $J_{n+1}(x)$ have no root in common save 0.

† Bourget, Ann. de l'École Normale, 1866, p. 66, stated such a theorem when n is integral without proof.

not both vanish for the same value of x. Let $x_0 \neq 0$ be a root of $J_{i+\frac{1}{2}}(x)$, and suppose that $g_i^{(i+\frac{1}{2})}(x_0) \neq 0$, we have

$$\tan x_0 = \frac{g_{i-1}^{(i+\frac{1}{2})}(x_0)}{g_i^{(i+\frac{1}{2})}(x_0)} = a, \tag{III}$$

where, if x_0 is merely algebraically irrational so is a. III becomes in terms of exponentials

$$e^{2x_0 i} = \frac{1+ai}{1-ai}.$$

Lindemann's equation,* from which he deduced the transcendental irrationality of π, asserts that if x_0 is algebraically irrational, this equation is impossible. Thus: *No two Bessel's functions whose orders are the halves of odd integers can have a positive root in common.*

The recurring relation connecting three consecutive g-functions suggests† a useful expression of $g_\kappa^{(n)}(x)/g_{\kappa-1}^{(n)}(x)$ as a continued fraction.

$$g_{i-1}^{(n)}(x)/g_{i-2}^{(n)}(x) = \frac{2(n+1)}{x} - \cfrac{1}{\cfrac{2(n+2)}{x} - \cdots - \cfrac{1}{\cfrac{2(n+i-1)}{x}}}.$$

This is evidently the $i-1^{\text{st}}$ convergent of the continued fraction

$$\frac{J_n(x)}{J_{n+1}(x)} = \frac{2(n+1)}{x} - \cfrac{1}{\cfrac{2(n+2)}{x} - \cfrac{1}{\cfrac{2(n+3)}{x} - \text{etc.}}}$$

Writing

$$g_{i-1}^{(n)}(x)/g_{i-2}^{n}(x) = \frac{2(n+1)}{x} - \phi(x, n)$$

from the continued fraction itself, it is easily seen that if, when n and x are positive, we slightly increase n, x remaining constant, the term $\frac{2(n+1)}{x}$ increases while $\phi(x, n)$ decreases, thus: *when n increases, the positive roots of $g_{i-1}^{(n)}(x)$ increase, if $n > 0$.*

It is easily shown that the roots of $g_{i-1}^{(n)}(x)$, as i increases indefinitely, cluster‡ about the roots of $J_n(x)$, so that we get a new proof of the theorem§ that the positive roots of $J_n(x)$, $n > 0$, increase with n.

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* Math. Ann., Bd. 20, p. 224.

† Hurwitz, Math. Ann., Bd. 33, l. c.

‡ Hurwitz, Math. Ann., Bd. 33, pp. 250–52.

§ Quoted on p. 196.

On page 200 it was seen that the roots of $g_{i-1}^{(n)}(x)$ can be used to find those intervals determined by consecutive positive roots of $J_n(x)$ in which no root of $J_{n+i}(x)$ lay. From the expression for $g_{i-1}^{(n)}(x)$ on page 201 we see that $g_{i-1}^{(n)}(x)$ has at most $E\left(\frac{i-1}{2}\right)$ positive roots. It is shown (p. 212, Math. Bull., Mar. 1897) that, *if n is positive and $2p < \kappa \leqq 2p+2$ where p is any positive integer, in each of the intervals* bounded by successive positive roots of $J_n(x)$, there will be one and only one root of $J_{n+\kappa}(x)$, except in p of these intervals, in which there will be no root of $J_{n+\kappa}(x)$.* Thus in the case we have just been considering there will be $E\left(\frac{i-1}{2}\right)$ vacant intervals, and consequently, *if n and i are positive, all the roots of $g_{i-1}^{(n)}(x)$ are real and, moreover, at most one lies in any interval determined by two consecutive positive roots of $J_n(x)$.*† When both n and i are negative we have a similar problem and one whose solution is deducible directly from the case in which n and i are positive. We have

$$\begin{aligned}
J_{n+i} &= \frac{2(n+i+1)}{x} J_{n+i+1} - J_{n+i+2}, \\
J_{n+i+1} &= \frac{2(n+i+2)}{x} J_{n+i+2} - J_{n+i+3}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
J_{n-2} &= \frac{2(n-1)}{x} J_{n-1} - J_n,
\end{aligned}$$

where both n and i are negative.

Eliminating as in the case where both n and i are positive, we get

$$J_{n+i}(x) = \bar{g}_{i-1}^{(n)}(x)\, J_{n-1}(x) - \bar{g}_{i-2}^{(n)}(x)\, J_n(x)$$

and readily obtain the relation

$$\bar{g}_i^{(n)}(x) = (-1)^i g_i^{(n)}(x).$$

Thus since the roots of $g_{i-1}^{(n)}(x)$ are all real, we have *the roots of $\bar{g}_i^{(n)}(x)$ are all real.* If n is an integer we have $J_{n+i}(x) = (-1)^{n+i} J_{-n-i}(x)$ and $J_n(x) = (-1)^n J_{-n}(x)$, so that there are $E\left(\frac{-i-1}{2}\right)$ of the intervals determined by the consecutive

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* When $J_{n+\kappa}(x)$ and $J_n(x)$ have a common root it may be regarded as lying in either of the intervals abutting on the common root.

† Hurwitz shows by a different method, p. 255, l. c. Math. Ann., Bd. 33, that when $n > 0$ all the roots of $g_{i-1}^{(n)}(x)$ are real, but does not show that not more than *one* root of $g_{i-1}^{(n)}(x)$ can lie in the interval delimited by two consecutive roots of $J_n(x)$.

positive roots of $J_n(x)$ in which there is one root of $g_{i-1}^{(n)}(x)$ and no root of $J_{n+i}(x)$. When n is not integral it is only necessary to let n increase to the value $-E(-n)$. We know that before n began to increase there were at most $E\left(\frac{-i-1}{2}\right)$ vacant intervals to the right of the origin, and when n reaches the value $-E(-n)$ we have seen that there are exactly $E\left(\frac{-i-1}{2}\right)$ such vacant intervals. It is, moreover, clear that when n increased to the value $-E(-n)$, no intervals were gained since the large roots of $J_{n+i}(x)$ and $J_n(x)$ all move out by equal amounts. Thus: *in each of the intervals delimited by the consecutive positive roots of* $J_n(x)$, $n<0$, *lies one and but one root of* $J_{n+i}(x)$, $i<0$, *except in* $E\left(\frac{-i-1}{2}\right)$ *of these intervals in which no root of* $J_{n+i}(x)$ *lies.*

The method hitherto employed does not enable us to treat the question of the reality of the roots of $g_i^{(n)}(x)$ when n and i have different signs. Here, in general, as the method of Hurwitz* readily shows, imaginary roots present themselves. The analytic expression for $g_{i-1}^{(n)}(x)$ on page 201 shows, however, that if $i=2\varkappa+1>0$ and $n<0$; if $\varkappa<-n<\varkappa+1$, all the roots of $g_{i-1}^{(n)}(x)$ are imaginary $\left(\text{all the coefficients of powers of } \frac{2}{x} \text{ being of same sign}\right)$, so that *the positive roots of* $J_{n+i}(x)$ $(n<0$ *and* i *odd) and* $J_n(x)$ *separate each other, if* $\frac{i-1}{2}<-n<\frac{i+1}{2}$ *or when* i *is even when* $\frac{i-2}{2}<-n<\frac{i+2}{2}$.

§5.—*On the Complex and Pure Imaginary Roots of* $J_n(x)$.

Hurwitz, in his paper "Ueber die Nullstellen der Bessel'schen Function," which has already been referred to, first enumerated the imaginary roots of $J_n(x)$, when n is real,† and determined the regions of the complex plane in which these roots lie both when n is real and when n is pure imaginary.

The theorem on page 197 affords an easy means of solving the first of these questions, and it is to this that we shall next turn our attention.

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* Hurwitz, Math. Ann., Bd. 33, l. c.

† It has long been known that when $n \geqq -1$, all the roots of $J_n(x)$ are real; the proof follows at once by the method due to Poisson (Theorie de la Chaleur, p. 178) from the integral

$$\int_0^a \nu J_n(\mu_\kappa \nu) J_n(\mu_\iota \nu)\, d\nu = \frac{a}{\mu_\kappa^2-\mu_\iota^2}\left[\mu_\kappa J_n(\mu_\iota a) J_{n+1}(\mu_\kappa a) - \mu_\iota J_n(\mu_\kappa a) J_{n+1}(\mu_\iota a)\right].$$

The imaginary roots of

$$J_n(x) = \frac{1}{\Gamma(1+n)}\left(\frac{x}{2}\right)^n\left[1 - \frac{1}{1+n}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,(1+n)(2+n)}\left(\frac{x}{2}\right)^4 - \text{etc.}\right]$$

are the same as the imaginary roots of

$$f(x, n) = \frac{1}{\Gamma(1+n)}\left[1 - \frac{1}{1+n}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,(1+n)(2+n)}\left(\frac{x}{2}\right)^4 - \right]$$
$$= \frac{1}{\Gamma(1+n)}\,\phi(x, n),$$

where it is to be noted that $f(x, n)$, regarded as a function of the two complex arguments x and n, has the following important properties:

If S denote a piece of the finite region of the x-planes, and Z a piece of the finite region of the n-planes.

1°. For any given value of n in the region Z, $f(x, n)$ is a finite, continuous, and singly-valued analytic function of x throughout S.

2°. For any value of x in the region S, $f(x, n)$ is a finite, continuous, and singly-valued analytic function of n throughout Z.

To prove these properties of $f(x, n)$ it is only necessary to note that for values of n in the neighborhood of a real negative integer $-i$, we have

$$\frac{1}{\Gamma(1+n)} = (n+i)\,\omega_1(n+i),$$

where $\omega_1(n+i)$ is analytic at $n = -i$, while $\phi(x_1, n) = \frac{1}{n+i}\,\omega_2(x_1, n+i)$, where ω_2 is analytic at $n = -i$; so that, since the product of two analytic functions is analytic, $f(x, n)$ is an analytic function of either x or n in the regions in question.

Concerning functions of two arguments, which satisfy the conditions 1° and 2°, the following general theorem* holds:

If $f(x, n) = 0$, *where* n *is a point within the region* Z, *has no roots on the boundary of the region* S; *the number*† *of roots of* $f(x, n)$ *within* S *will remain unchanged, if* n *be changed to* $n + \Delta n$, *where* $|\Delta n| < \rho$ *and where* ρ *is a sufficiently small positive quantity.*

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* See, for instance, Neumann's Abelsche Integrale, p. 141.

† A κ-fold root is, of course, to be counted as κ simple roots.

Let us now suppose that n, starting with a value $-\frac{1}{2}$, in which case, since

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \cos x,$$

all the roots of $f(x, n)$ are real, decrease through the value -1. All the roots of $f(x, n)$ are moving towards the origin, and when n passes through the value -1, the two that reach the origin when $n = -1$, disappear from the axis of reals and must, therefore, by the theorem just quoted, become conjugate imaginary. In the same way, when n continually decreasing passes through the value -2, two more roots become conjugate imaginary. When n decreases to the value $\bar{n}$ (not integral), $2E(-\bar{n})$ roots become conjugate imaginary by disappearing from the axis of reals at the origin. Since $J_n(x)$* can have no multiple roots save at $x = 0$, all the imaginary roots of $f(x, n)$ must arise at the origin and at the point ∞. It is easy to show, however, by means of the asymptotic value of $J_n(x)$ that no imaginary roots can come in from the point ∞. In the region of the x-plane to the right of the axis of pure imaginaries, we have asymptotically†

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \cos\left\{\frac{2n+1}{4}\pi - x\right\},$$

and along the axis of pure imaginaries,‡

$$J_n(x) = i^n \sqrt{\frac{i}{2\pi x}}\, e^{-xi}.$$

Since, if x_0 be a root of $J_n(x)$ so also is $-x_0$, we need only consider the quarter of the plane bounded by the positive halves of the axes of reals and pure imaginaries, and here the asymptotic values given above show that all the large roots of $J_n(x)$ are real.

The considerations on pages 197-8 show that when $-1 \gtreqless n \gtreqless -\frac{1}{2}$, all the roots of $J_n(x)$ are real. If then we let n, starting with a value a little greater than -1, increase, all the roots of $f(x, n)$ will move away from the origin, and as no roots can disappear from the axis of real, we have the theorem:

Except when n is a negative integer, $J_n(x)$ has $2E(-n)$ conjugate imaginary roots, together with an infinite number of real roots.

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* $J_n(x)$ and $f(x, n)$ have the same roots save at the origin.

† Gray and Mathews, Bessel Functions, p. 70, and Jordan, Cours d'Analyse, tom. 3, p. 265, §212 (1895).

‡ Gray and Mathews, p. 68.

As n approaches a negative *integral* value, the moduli of all the imaginary roots become indefinitely small, since, when n is integral, we have $J_n(x) = (-1)^n J_{-n}(x)$, so that here all the roots are real.

Since the roots of $J_n(x)$ are symmetrical with respect to the origin and the axis of reals, if $E(-n)$ is odd, there must be at least two pure imaginary roots. Hurwitz shows, and the result could be deduced by the considerations we have employed, using $-\left(\frac{x}{2}\right)^2$ instead of x as the independent variable, that when $E(-n)$ is odd there are *only* two pure imaginary roots, while if $E(-n)$ is even, all the roots are complex.

Hitherto we have considered only questions relating to the roots of the Bessel's functions and the convergents of the continued fraction for J_n/J_{n+1}. The same methods are, of course, applicable to functions defined by any homogeneous linear differential equation of the second order; but as the variable parameters become more numerous, the theorems obtained become more special and the conditions under which they hold more restricted.

In the case of the Bessel's functions, the asymptotic value of the solution corresponding to the lesser exponent enabled us to determine the motion of the roots as n decreased. This method is, of course, not applicable to the corresponding problem in the case of the hypergeometric functions with which the second part of this paper deals.

II.

ON THE ROOTS OF THE HYPERGEOMETRIC SERIES.

§6.—*On the Variation of the Roots of $F(\alpha, \beta, \gamma, x)$ with the Parameters.*

The hypergeometric series satisfies the differential equation

$$\frac{d^2y}{dx^2} + \frac{\gamma - (\alpha + \beta + 1)x}{x(1-x)} \frac{dy}{dx} - \frac{\alpha\beta}{x(1-x)} y = 0, \tag{1}$$

where the singular points are

$$0 \qquad \infty \qquad \text{and} \qquad 1,$$

and the corresponding exponents are

$$0,\ 1-\gamma; \quad \alpha,\ \beta; \quad 0,\ \gamma - \alpha - \beta.$$

The exponent-differences being

$$\lambda = \gamma - 1, \quad \mu = \alpha - \beta, \quad \nu = \alpha + \beta - \gamma,$$

where it is supposed that α, β, and γ are real.

By the change of dependent variable

$$y = \bar{y}\, x^{-\frac{1+\lambda}{2}} (1 - x)^{-\frac{1+\nu}{2}},$$

(1) becomes

$$\frac{d^2\bar{y}}{dx^2} = \phi(\lambda, \mu, \nu, x)\,\bar{y}$$
$$= -\tfrac{1}{4}\, \frac{x^2(1-\mu^2) + x(\mu^2 - \nu^2 + \lambda^2 - 1) + 1 - \lambda^2}{x^2(1-x)^2}\,\bar{y}.$$

When λ is positive, and in the present section we shall always suppose that this is the case, the solution corresponding to the larger exponent of $x = 0$ is the ordinary hypergeometric series

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha \,.\, \beta}{1!\,\gamma}\, x + \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{2!\,\gamma(\gamma+1)}\, x^2 +,$$

where, since α and β are interchangeable, we may always assume

$$\mu = \alpha - \beta \geqq 0.$$

Moreover, since

$$\frac{\partial \phi(\lambda, \mu, \nu, x)}{\partial \mu} = -\tfrac{1}{2}\, \frac{\mu}{x(1-x)},$$

we see that ϕ decreases with the increase of μ for all values of x between 0 and 1.

By III′ we see that if $x_r = \phi_r(\mu)$ denote the r^{th} root of $y(x, \mu) = F(\alpha, \beta, \gamma, x)$ between 0 and 1, x_r will *increase* as μ *decreases*. III′ is, of course, not applicable in the case $\lambda < 0$, and as yet I have found no way of treating the variation of the roots of a solution corresponding to the lesser exponent.

The next question that we shall consider is the following: If μ decrease by a small amount, $\Delta\mu$, λ and ν remaining constant, we have seen that all the roots of

$$y(x, \mu) = F(\alpha, \beta, \gamma, x)$$

between 0 and 1 are slightly increased and there will therefore be one and but one root of $y(x, \mu - \Delta\mu)$ in each of the intervals x_1x_2, x_2x_3, ..., x_ω*1 determined by the roots of $y(x, \mu)$ between 0 and 1. The question is, how large

* Using x_ω to denote the root of $y(x\ \mu,)$ which in interval 01 is nearest 1.

may $\Delta\mu$ become and $y(x, \mu - \Delta\mu)$ still have one root in each of these intervals. To answer this question it is necessary to have recourse to the explicit expression of $F(\alpha, \beta, \gamma, x)$ in terms of the two (in general) linearly independent solutions about $x = 1$. We have

$$F(\alpha, \beta, \gamma, x) = aF(\alpha, \beta, \alpha + \beta - \gamma + 1, 1 - x) \\ + b(1-x)^{-\nu} F(\gamma - \beta, \gamma - \alpha, \gamma - \alpha - \beta + 1, 1 - x), \quad (2)$$

where*

$$a = \frac{\Gamma(1+\lambda)\Gamma(-\nu)}{\Gamma\left(\frac{1+\lambda-\mu-\nu}{2}\right)\Gamma\left(\frac{1+\lambda+\mu-\nu}{2}\right)},$$

$$b = \frac{\Gamma(1+\lambda)\Gamma(2+\nu)}{\Gamma\left(\frac{1+\lambda+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\lambda-\mu+\nu}{2}\right)}$$

expressed in terms of exponent differences

$$F(\alpha, \beta, \gamma, x) = F\left(\frac{1+\lambda+\mu+\nu}{2}, \frac{1+\lambda-\mu+\nu}{2}, 1+\lambda, x\right).$$

If λ and ν are positive, all the terms of this series are positive, and it can have no positive roots unless

$$\mu > 1 + \lambda + \nu.$$

Let us suppose that μ, starting with a value $\bar{\mu}$, decrease; all the roots of $F(\alpha, \beta, \gamma, x)$ in the interval 01 will increase, and in each of the intervals determined by the consecutive roots of $y(x, \bar{\mu})$ which lie between 0 and 1, there will be one root of $y(x, \mu - \Delta\mu)$, if $\Delta\mu$ is sufficiently small.

It is clear by III′ that this state of affairs will not be altered until the root x_ω reaches the point 1, thus the roots are lost one at a time as μ decreases, disappearing from the axis of reals at the point $x = 1$. To determine when the root x_ω reaches the point 1 we have merely to refer to (2), above.

Since $\nu > 0$, the sign of $F(\alpha, \beta, \gamma, x)$, for values of x a little less than 1, depends only on b, and therefore only on $\Gamma\left(\frac{1+\lambda-\mu+\nu}{2}\right) = \Gamma(\beta)$. When, therefore, β, by the decrease of μ, increases through the next negative integer, the Γ-function changes sign and the root x_ω must have moved up to the point 1, so when β, continually increasing as μ decreases, passes through the *next* larger negative integer, the root $x_{\omega-1}$ passes into the point 1 and we have the

*Gauss, Ges. Werke, Bd. III, pp. 211–13.

theorem: *When* $\nu > 0$, $F(\alpha - \varkappa, \beta + \varkappa, \gamma, x)$ *will have one and but one root in each of the intervals* x_1x_2, x_2x_3, ..., $x_\omega 1$, *determined by the roots of* $F(\alpha, \beta, \gamma, x)$, *if* $0 < \varkappa \leqq -\beta - E(-\beta)$ *the extremity of the interval* $x_\omega 1$ *being included.* Or more generally: $F(\alpha - \varkappa, \beta + \varkappa, \gamma, x)$ *will have one and but one root in each* of the intervals* x_1x_2, x_2n_3 .... $x_\omega 1$ *except in* i *of these interval, in which no root of* $F(\alpha - \varkappa, \beta + \varkappa, \gamma, x)$ *lies, if*

$$\eta + i - 1 < \varkappa \leqq \eta + i; \quad \beta + \varkappa < 0,$$

when $\eta = -\beta - E(-\beta)$.

When $\nu < 0$ (we assume that λ and μ are always positive), since

$$F(\alpha, \beta, \gamma, x) = (1 - x)^{-\nu} F(\gamma - \beta, \gamma - \alpha, \gamma, x),$$

and since $F(\gamma - \beta, \gamma - \alpha, \gamma, x)$ satisfies a differential equation with the exponent differences λ, μ, $-\nu$, we have the theorem: *When* $\nu < 0$, $F(\alpha - \varkappa, \beta + \varkappa, \gamma, x)$ *has one and but one root in each of the intervals* x_1x_2, x_2x_3 .... $x_\omega 1$, *except in* i *of these intervals in which there is no root of* $F(\alpha - \varkappa, \beta + \varkappa, \gamma, x)$ *if*

$$\eta + i - 1 < \varkappa \leqq \eta + i,$$

where $$\eta = \alpha - \gamma - E(\alpha - \gamma) \text{ and } \alpha - \gamma - \varkappa > 0.$$

In precisely the same way, since

1°. $$\frac{\partial \phi}{\partial \lambda} = \frac{1}{2} \frac{\lambda}{x^2(1 - x)} > 0, \quad 0 < x < 1,$$

if $\lambda > 0$.

2°. $$\frac{\partial \phi}{\partial \nu} = \frac{1}{2} \frac{\nu}{x(1 - x)^2} > 0, \quad 0 < x < 1,$$

if $\nu > 0$.

By 1° $F(\alpha + \varkappa, \beta + \varkappa, \gamma + 2\varkappa, x)$ *will have one and but one root in each of the intervals* x_1x_2, ..., $x_\omega 1$, *delimited by the roots of* $F(\alpha, \beta, \gamma, x)$, *save* i, *in which there will be no root of* $F(\alpha + \varkappa, \beta + \varkappa, \gamma + 2\varkappa, x)$ *if*

$$\eta + i - 1 < \varkappa \leqq \eta + i \text{ and } \nu > 0,$$

when, as before, $\eta = -\beta - E(-\beta)$ and $\beta + \varkappa < 0$, or if $\nu < 0$, $\eta = \alpha - \gamma - E(\alpha - \gamma)$ and $\alpha - \gamma - \varkappa > 0$.

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* When $F(\alpha, \beta, \gamma, x)$ and $F(\alpha - \kappa, \beta + \kappa, \gamma, x)$ have a common root, it may be regarded as lying in either of the intervals separated by the common root.

In both cases we have, of course, the restriction

$$1 - \gamma - 2\varkappa \leqq 0.$$

By 2° we have in the same way, *when* $\nu > 0$, $F(\alpha + \varkappa,\ \beta + \varkappa,\ \gamma,\ x)$, *have one and but one root in each of the intervals* $x_1 x_2$, $x_2 x_3 \ldots\ldots x_\omega 1$, *save in* i *of these intervals in which there will be no root of* $F(\alpha + \varkappa,\ \beta + \varkappa,\ \gamma,\ x)$ *when*

$$\eta + i - 1 < \varkappa \leqq \eta + i \text{ and } \beta + \varkappa < 0,$$

or if $\nu < 0$, *there will be* i *intervals containing no roots of* $F(\alpha - \varkappa,\ \beta - \varkappa,\ \gamma,\ x)$ *if*

$$\eta + i - 1 < \varkappa \leqq \eta + i, \quad \alpha - \gamma - \varkappa > 0,$$

where
$$\eta = \alpha - \gamma - E(\alpha - \gamma).$$

These theorems are analogous to the theorem for the Bessel's function quoted on page 204.

The theorem that we have just obtained might be used to get certain results concerning the separation of the roots of a hypergeometric function by those of a contiguous function, but the general* theorem is so much more easily established by the methods employed by Van Vleck that it is not worth while to consider them here. We need only note that as in the case of the kindred Bessel's functions, if F and F_κ denote two kindred hypergeometric series, the vacant intervals which the considerations of the previous pages show must, in general, arise, are determined by the vanishing of a convergent of a continued fraction, which, in the chain of contiguous functions,

$$F,\ F_1,\ F_2,\ \ldots\ldots F_{\kappa-1},\ F_\kappa, \text{ is either } F/F_1 \text{ or } F_\kappa/F_{\kappa-1}.$$

The considerations of page 210 admit of an immediate application to the question of the enumeration of the roots of $F(\alpha,\ \beta,\ \gamma,\ x)$ between 0 and 1. The word root will, in the next section, be used to denote a real root.

§7.—*On the Number of Roots of the Hypergeometric Series between* 0 *and* 1.†

This problem, of which Klein‡ first published a solution in 1890, depending on the conformal property of the Schwarzian s-function and a discussion of the

* The theorem referred to is due to Van Vleck (Amer. Jour. XIX, l. c.); it may be thus stated: *In an interval of the x-axis containing no singular points of the differential equation, the real roots of two contiguous hypergeometric functions separate each other.*

† The solution here given is, with slight alterations, the same as that published in my paper in the Math. Bull., May, 1897. ‡ Math. Ann., Bd. 37.

shape of the circular triangles on which the x-halfplane is mapped, has been solved more recently by Hurwitz* and Gegenbauer.† The solutions of Hurwitz and Gegenbauer both depend on the determination of a chain of contiguous hypergeometric functions which can be used as a set of Sturmian functions. Klein's method, on the contrary, only makes use of the differential equation, but while extremely elegant and interesting, does not lead to this result so directly as the methods of Sturm which we have been employing.

On page 210 we saw that when $\lambda > 0$ and $\nu > 0$, one root in the interval 01 was lost whenever β, through the decrease of μ, increased through a negative integer. When β, always increasing, reaches a value a little greater than zero, $\overline{E}(1-\beta)$‡ roots will have been lost, and as the hypergeometric series then has all its terms positive and can have no roots > 0, all the roots between 0 and 1 must have been lost. Thus when $\lambda > 0$ *and* $\nu > 0$, *the hypergeometric series has* $\overline{E}(1-\beta)$ *roots between* 0 *and* 1, and in the same way by the considerations on page 211 when $\lambda > 0$ *and* $\nu < 0$ *the hypergeometric series has* $\overline{E}(1-(\gamma-\alpha))$ *roots between* 0 *and* 1.

When $\lambda < 0$, we must take, not the hypergeometric series, but

$$y = x^{-\lambda} F\left(\frac{1-\lambda+\mu+\nu}{2}, \quad \frac{1-\lambda-\mu+\nu}{2}, \quad 1-\lambda, \quad x\right),$$

which is then the solution corresponding to the larger exponent of the origin. The hypergeometric series on the right which we may denote by $F(\alpha', \beta', \gamma', x)$ satisfies a differential equation whose exponent differences are $-\lambda, \mu, \nu$, so that *in the interval* 01 $F(\alpha', \beta', \gamma', x)$ *has* $\overline{E}(1-\beta')$ roots if $\nu > 0$ and $\overline{E}[1-(\gamma'-\alpha')]$ roots if $\nu < 0$.

All four results can be stated in the one formula

$$\chi = \overline{E}\left(\frac{\mu - |\lambda| - |\nu| + 1}{2}\right).$$

When $\lambda < 0$, we determined not the number of roots of $F(\alpha, \beta, \gamma, x)$ between 0 and 1, but the number of roots of a solution which, in general, is linearly independent of $F(\alpha, \beta, \gamma, x)$.

Theorem I′ of the introduction shows that in this case $F(\alpha, \beta, \gamma, x)$ has $N = \chi$ or $\chi + 1$ roots between 0 and 1, the even or odd value of N being chosen according as $y_{x=1}$ is $>$ or < 0. There are thus two cases to be considered.

* Math. Ann., Bd. 38. † Sitz. Bericht., Wien. Akad., Bd. 100, 2*a*.

‡ $\overline{E}(s)$ denotes the largest positive integer less than s, while $\overline{E}(s) = 0$ if $s \leq 1$.

1°. $\nu > 0$. Here, according as b is $>$ or < 0, i. e. according as

$$\Gamma(1+\lambda)\Gamma\left(\frac{1+\lambda+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\lambda-\mu+\nu}{2}\right) \text{ is } > \text{ or } < 0,$$

the sign of $y_{x=1}$ is $>$ or < 0.

2°. $\nu < 0$. Here, according as a is $>$ or < 0, i. e. according as

$$\Gamma(1+\lambda)\Gamma\left(\frac{1+\lambda-\mu-\nu}{2}\right)\Gamma\left(\frac{1+\lambda+\mu-\nu}{2}\right) \text{ is } > \text{ or } < 0,$$

$y_{x=1}$ is $>$ or < 0.

If in 1° $b=0$, i. e. if $(1+\lambda+\mu+\nu)/2$ or $(1+\lambda-\mu+\nu)/2$ is zero or a negative integer, $F(\alpha, \beta, \gamma, x)$ ceases to be linearly independent of $F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)$, which, when $\nu > 0$, is the solution corresponding to the larger exponent of $x=1$, so that by the theorem I′ of Sturm $F(\alpha, \beta, \gamma, x)$ has in this case χ roots between 0 and 1.

For the same reason in 2°, when $a=0$, i. e. when $(1+\lambda-\mu-\nu)/2$ or $(1+\lambda+\mu-\nu)/2$ is zero or a negative integer, the hypergeometric series has χ roots between 0 and 1.

It will be noticed that in discussing the general problem of enumerating the roots of $F(\alpha, \beta, \gamma, x)$, we have been led naturally to consider the case where $\beta=0$ or a negative integer, and have in a certain sense generalized the results of Stieltjes, Comptes Rendus, tom. 100, and Hilbert, Crelle, Bd. 103, which refer to this case.

HARVARD UNIVERSITY, *June* 3, 1897.